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Asymptotic Transfer Function Analysis of Conical Shells

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An asymptotic transfer function method is presented for modeling and analysis of conical shells. The displacement functions are first expanded in Fourier series in the circumferential direction, and the motion equations are decoupled into a group of partial differential equations with one space variable and one time variable. Introducing a small perturbation parameter and using the Laplace transformation and perturbation technique, the partial differential equations with variable coefficients are reduced to ordinary differential equations with constant coefficients, which are solved by the transfer function method. The method is used to perform analysis of stepped conical shells with different conical angle or thickness and subjected to various initial and boundary conditions. Numerical methods are presented and compared with the finite element method.

Introduction

CONICAL shells have wide applications in aeronautic, astronautic, civil, and chemical engineering. The research on their mechanical behavior under various external excitations and boundary restrictions has great importance in engineering practice. As one type of revolutionary thin shells, conical shells have been studied by many researchers, and a lot of modeling and analysis methods have been developed. Chang[1] gave a literature review of the vibration of conical shells. Liew[2] reviewed recent developments in the free vibration analysis of thin, moderately thick shallow shells. Compared with cylindrical shells, conical shells are difficult to analyze in exact and closed form because of the mathematical complexity in geometry and variable surface curvature.[2] Wan[3,4] obtained a closed-form solution of the variable coefficient differential equations of conical shells in terms of generalized hypergeometric functions. Tong[5] obtained the solution of laminated conical shells in the form of power series. However, their solutions are very complicated and are difficult to use for complex loads, boundary conditions, and geometric configurations. Therefore, approximate or numerical methods, such as Raleigh–Ritz, Galerkin, finite difference, and finite element methods, have been widely used in the analyses. Teichmann[6] presented an approximate solution of fundamental frequencies and buckling loads of cylindrical and conical shell panels. Srinivasan and Krishnan[7] provided the free vibration frequencies of fully clamped open conical shells by using an integral equation approach. Cheung et al.[8] employed a spline finite strip method to investigate the natural frequencies of fully clamped singly curved shells, and design charts for specific fully supported conical shell configurations were presented. Xi et al.[9] studied free vibration of composite shells of revolution by the finite element method. Sivadas and Ganesan[10] conducted vibration analysis of laminated conical shells with variable thickness. These methods provide effective ways for engineering analysis in most cases. However, their defaults are obvious in some specific situations, such as analysis concerned with stress concentration, high-frequency response, etc.

Based on the method proposed in Refs. 11 and 12, an asymptotic distributed transfer function method for the analysis of conical shells is presented in this paper. First, the displacements, external excitations, and boundary conditions are expanded in Fourier series in a circumferential direction. Because of the orthogonality of trigonometric functions, the governing equations for different wave numbers are decoupled and Laplace transformation is used to transform the time t to obtain ordinary differential equations with complex parameter s. Second, introducing the perturbation parameter $\varepsilon = L\sin\alpha/r_0$, those ordinary differential equations with variable coefficients are reduced to a group of ordinary differential equations with constant coefficients. Third, casting them into state-space form, the solutions of boundary value problems are given by transfer function formulation in closed form. Last, assembling techniques are introduced to solve complex combined conical shells.

The advantages of the proposed method are threefold. First, it provides closed-form asymptotic solutions with very high precision for various static, dynamic, and control problems of conical shells. Second, it treats different shell models (Love–Timoshenko, Donnell–Mushtari, or Koiter–Sanders type, elastic or viscoelastic, stationary or spinning shells, etc.), arbitrary initial and boundary conditions, and external excitations in a simple and unified manner so that the computer coding of the method is easy. Third, it is capable of synthesizing combined conical shells in which each subshell section may have a different conical angle and thickness, as shown in Fig. 1. These advantages will be demonstrated in the following analysis and numerical simulations.

General Formulations

In this section, the response of a homogeneous conical shell with constant conical angle and thickness is considered. The shell geometry and coordinate definition are shown in Fig. 2. In the most general form, the dynamic response of the shell is governed by following partial differential equations:

$$\sum_{k=1}^{3}\sum_{i=0}^{n_k}\sum_{j=0}^{i}\left[A_{mkij}(x)\frac{\partial^2}{\partial t^2} + B_{mkij}(x)\frac{\partial}{\partial t} + C_{mkij}(x)\right] \times \frac{\partial^i u^k(x,\theta,t)}{\partial x^{i-j}\partial\theta^j} = f^m(x,\theta,t) \qquad (m = 1,2,3) \tag{1}$$

where u^k $(k = 1, 2, 3)$ are the shell displacements in the generatrix (x-), circumferential (θ-), and normal (z-) coordinate directions, respectively; n_k is the highest order of differentiation of u^k; $A_{mkij}(x)$, $B_{mkij}(x)$, and $C_{mkij}(x)$ are known functions; and $f^m(x,\theta,t)$ $(m = 1, 2, 3)$ are the external excitations.

The boundary and initial conditions of the shell are

$$\left[\sum_{k=1}^{3}\sum_{i=0}^{n_k-1}\sum_{j=0}^{i}\beta_{kij}^{l_m}\frac{\partial^i u^k(x,\theta,t)}{\partial x^{i-j}\partial\theta^j}\right]_{x=x_m} = \lambda_m^{l_m}(\theta,t) \qquad (m = 1,2,\ l_m = 1,2,\ldots,n_{b,m}) \tag{2a}$$

$$u^k(x,\theta,t)\big|_{t=0} = u_0^k(x,\theta) \qquad (k = 1,2,3) \tag{2b}$$

$$\frac{\partial}{\partial t}u^k(x,\theta,t)\bigg|_{t=0} = u_{t,0}^k(x,\theta) \qquad (k = 1,2,3) \tag{2c}$$

where x_1 and x_2 are the generatrix coordinate values of the left and right ends of the shell, respectively; $\beta_{kij}^{l_m}$ are constants; and $\lambda_m^{l_m}(\theta,t)$, $u_0^k(x,\theta)$, and $u_{t,0}^k(x,\theta)$ are given functions representing the boundary and initial disturbances. The number of boundary conditions,

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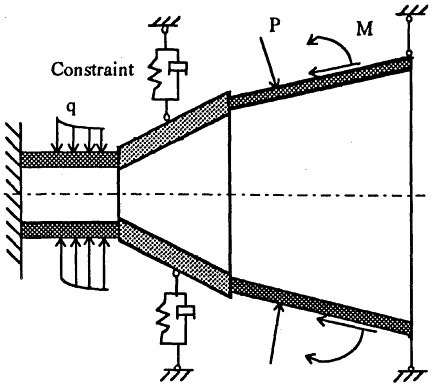

Fig. 1 Constrained/combined conical shell.

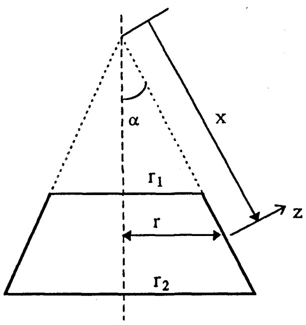

Fig. 2 Parameter and coordinate definition of a conical shell.

i.e., the sum of $n_{b,1}$ and $n_{b,2}$, satisfies $n_{b,1} + n_{b,2} = n_1 + n_2 + n_3$ and is eight in most situations. Equations (1) and (2) represent many linear shell models, such as Love–Timoshenko, Donnell–Mushtari, Flügge–Novozhilov, Reissner, Vlasov, and Koiter–Sanders, etc.

To find the solution of the initial boundary value problem given by Eqs. (1) and (2), the displacements and all given functions in Eqs. (1) and (2) are expanded into Fourier series, as was done by Zhou and Yang[11] and Yang and Zhou.[12] For example, the displacements are written as

$$u^k(x,\theta,t) = \sum_{n=0}^{\infty}\left[u^k_{1,n}(x,t)\cos n\theta + u^k_{2,n}(x,t)\sin n\theta\right] \qquad (k = 1,3) \quad \text{(3a)}$$

$$u^2(x,\theta,t) = \sum_{n=0}^{\infty}\left[u^2_{1,n}(x,t)\sin n\theta + u^2_{2,n}(x,t)\cos n\theta\right] \quad \text{(3b)}$$

Substituting these series into Eqs. (1) and (2) and noticing the orthogonality of trigonometric functions, it is easy to show that the motion equations and initial and boundary conditions for different wave numbers n are decoupled. Applying the Laplace transform to these decoupled equations and casting them in state-space form lead to

$$\frac{\mathrm{d}\boldsymbol{\eta}_n(x,s)}{\mathrm{d}x} = \boldsymbol{F}_n(x,s)\cdot\boldsymbol{\eta}_n(x,s) + \tilde{\boldsymbol{f}}_n(x,s) + \tilde{\boldsymbol{g}}_n(x,s) \qquad (n = 0,1,2,\ldots) \quad (4)$$

where the state-space vector $\boldsymbol{\eta}_n$ contains the displacement function $\tilde{u}^k_{i,n}$, with the tilde denoting the Laplace transform of the corresponding function,

$$\boldsymbol{\eta}_n = \left\{\boldsymbol{\eta}^T_{1,1,n} \quad \boldsymbol{\eta}^T_{2,1,n} \quad \boldsymbol{\eta}^T_{3,1,n} \quad \boldsymbol{\eta}^T_{1,2,n} \quad \boldsymbol{\eta}^T_{2,2,n} \quad \boldsymbol{\eta}^T_{3,2,n}\right\}^T \in C^{n_{b,1}+n_{b,2}} \quad \text{(5a)}$$

$$\boldsymbol{\eta}_{k,i,n} = \left\{\tilde{u}^k_{i,n} \quad \frac{\mathrm{d}\tilde{u}^k_{i,n}}{\mathrm{d}x} \quad \cdots \quad \frac{\mathrm{d}^{n_k-1}\tilde{u}^k_{i,n}}{\mathrm{d}x^{n_k-1}}\right\}^T \in C^{n_k} \qquad (k = 1,2,3,\ i = 1,2) \quad \text{(5b)}$$

$\boldsymbol{F}_n(x,s)$ is an $(n_{b,1}+n_{b,2}) \times (n_{b,1}+n_{b,2})$ complex matrix containing the coefficients $A_{mkij}(x)$, $B_{mkij}(x)$, and $C_{mkij}(x)$ of Eq. (1), and the complex $(n_{b,1}+n_{b,2})$ vectors $\tilde{\boldsymbol{f}}_n(x,s)$ and $\tilde{\boldsymbol{g}}_n(x,s)$ are composed of Laplace transforms of the Fourier expansion of external and initial disturbances, respectively. Similarly, the boundary conditions (2a) are cast into the matrix form

$$\boldsymbol{M}_n(s)\cdot\boldsymbol{\eta}_n(x_1,s) + \boldsymbol{N}_n(s)\cdot\boldsymbol{\eta}_n(x_2,s) = \boldsymbol{\gamma}_n(s) \quad (6)$$

where the boundary matrices

$$\boldsymbol{M}_n(s), \qquad \boldsymbol{N}_n(s)\left(\in C^{(n_{b,1}+n_{b,2})\times(n_{b,1}+n_{b,2})}\right)$$

contain the coefficients β^{lm}_{kij} of Eq. (2a) and the $(n_{b,1}+n_{b,2})$-vector $\boldsymbol{\gamma}_n(s)$ contains the Laplace transforms of the Fourier expansions of boundary excitation functions. For more detail, see Ref. 11.

Different from the case of cylindrical shells, the solution of Eqs. (4) and (6) cannot be obtained directly because $\boldsymbol{F}_n(x,s)$ is dependent on x.

Introduce the linear coordinate transform

$$\xi = \frac{1}{L}(x-\bar{x}), \qquad \bar{x} = \frac{x_1+x_2}{2}, \qquad -1 \le \xi \le 1 \quad \text{(7a)}$$

where L is the half-length of the shell generatrix. From Eq. (7a) and Fig. 2, it can be seen that

$$r = r_0(1+\varepsilon\xi) \quad \text{(7b)}$$

$$x = r_0(1+\varepsilon\xi)/\sin\alpha \quad \text{(7c)}$$

in which $r_0 = \bar{x}\sin\alpha$ and $\varepsilon = (L/r_0)\sin\alpha$. Obviously, when α or L/r_0 is small, ε is a suitable small parameter for perturbation.

Inserting Eq. (7) into Eqs. (4) and (6), we have

$$\frac{\mathrm{d}\boldsymbol{\eta}_n(\xi,s,\varepsilon)}{\mathrm{d}\xi} = \bar{\boldsymbol{F}}_n(\varepsilon\xi,s)\cdot\boldsymbol{\eta}_n(\xi,s,\varepsilon) + \boldsymbol{q}_n(\xi,s) \quad \text{(8a)}$$

$$\boldsymbol{M}_n(s)\cdot\boldsymbol{\eta}_n(-1,s,\varepsilon) + \boldsymbol{N}_n(s)\cdot\boldsymbol{\eta}_n(1,s,\varepsilon) = \boldsymbol{\gamma}_n(s) \quad \text{(8b)}$$

where

$$\boldsymbol{\eta}_n(\xi,s,\varepsilon) = \boldsymbol{\eta}_n(x,s), \qquad \bar{\boldsymbol{F}}_n(\varepsilon\xi,s) = L\boldsymbol{F}_n(x,s)$$

$$\boldsymbol{q}_n(\xi,s) = L[\tilde{\boldsymbol{f}}_n(x,s) + \tilde{\boldsymbol{g}}_n(x,s)]$$

Noticing Eqs. (7b) and (7c), the expansions of $\boldsymbol{\eta}_n(\xi,s,\varepsilon)$ and $\bar{\boldsymbol{F}}_n(\varepsilon\xi,s)$ in perturbation parameter ε are

$$\bar{\boldsymbol{F}}_n = \sum_{i=0}^{m}\varepsilon^i\xi^i\boldsymbol{F}^i_n(s) \quad \text{(9a)}$$

$$\boldsymbol{\eta}_n(\xi,s,\varepsilon) = \sum_{i=0}^{m}\varepsilon^i\boldsymbol{\eta}^i_n(\xi,s) \quad \text{(9b)}$$

where m is the perturbation order.

Substituting Eq. (9) into Eq. (8), the following linear ordinary differential equations with constant coefficients are obtained:

$$\begin{gathered}\frac{\mathrm{d}\boldsymbol{\eta}^0_n(\xi,s)}{\mathrm{d}\xi} = \boldsymbol{F}^0_n(s)\cdot\boldsymbol{\eta}^0_n(\xi,s) + \boldsymbol{q}_n(\xi,s) \\ \boldsymbol{M}_n(s)\cdot\boldsymbol{\eta}^0_n(-1,s) + \boldsymbol{N}_n(s)\cdot\boldsymbol{\eta}^0_n(1,s) = \boldsymbol{\gamma}_n(s)\end{gathered} \quad \text{(10a)}$$

$$\begin{gathered}\frac{\mathrm{d}\boldsymbol{\eta}^i_n(\xi,s)}{\mathrm{d}\xi} = \boldsymbol{F}^0_n(s)\cdot\boldsymbol{\eta}^i_n(\xi,s) + \sum_{j=1}^{i}\xi^j\boldsymbol{F}^j_n(s)\cdot\boldsymbol{\eta}^{i-j}_n(\xi,s) \qquad i = 1,2,\ldots \\ \boldsymbol{M}_n(s)\cdot\boldsymbol{\eta}^i_n(-1,s) + \boldsymbol{N}_n(s)\cdot\boldsymbol{\eta}^i_n(1,s) = 0\end{gathered} \quad \text{(10b)}$$

The solution of Eq. (10) is given by transfer function formulation

$$\boldsymbol{\eta}^0_n(\xi,s) = \int_{-1}^{1}\boldsymbol{G}_n(\xi,x,s)\cdot\boldsymbol{q}_n(x,s)\,\mathrm{d}x + \boldsymbol{H}_n(\xi,s)\cdot\boldsymbol{\gamma}_n(s) \quad \text{(11a)}$$

$$\boldsymbol{\eta}^i_n(\xi,s) = \int_{-1}^{1}\boldsymbol{G}_n(\xi,x,s)\cdot\sum_{j=1}^{i}x^j\boldsymbol{F}^j_n\cdot\boldsymbol{\eta}^{i-j}_n(x,s)\,\mathrm{d}x \quad \text{(11b)}$$

in which

$$\boldsymbol{G}_n(\xi,x,s) = \begin{cases}\boldsymbol{H}_n(\xi,s)\cdot\boldsymbol{M}(s)\cdot e^{-\boldsymbol{F}^0_n(1+x)} & x \le \xi \\ -\boldsymbol{H}_n(\xi,s)\cdot\boldsymbol{N}(s)\cdot e^{\boldsymbol{F}^0_n(1-x)} & x > \xi\end{cases} \quad \text{(11c)}$$

$$\boldsymbol{H}_n(\xi,s) = e^{\boldsymbol{F}^0_n(s)\xi}\cdot\left(\boldsymbol{M}_n(s)\cdot e^{-\boldsymbol{F}^0_n} + \boldsymbol{N}_n(s)\cdot e^{\boldsymbol{F}^0_n}\right)^{-1} \quad \text{(11d)}$$

Having η_n in hand, the displacements, strains, and stresses can be calculated easily. For static response, the complex variable $s = 0$. For frequency response, $s = i\omega$.

Eigenvalue Problem

For the analysis of eigenvalue problems of free vibration, letting $s = i\omega$, $F(\varepsilon\xi, i\omega)$ becomes

$$\bar{F}_n(\varepsilon\xi, i\omega) = \bar{F}_{n,c}(\varepsilon\xi) + \omega^2 \bar{F}_{n,\omega}(\varepsilon\xi) \tag{12}$$

Expanding $\bar{F}_{n,c}(\varepsilon\xi)$ and $\bar{F}_{n,\omega}(\varepsilon\xi)$ in perturbation parameter ε,

$$\bar{F}_{n,c} = F^0_{n,c} + \varepsilon\xi F^1_{n,c} + \varepsilon^2\xi^2 F^2_{n,c} + \cdots \tag{13a}$$

$$\bar{F}_{n,\omega} = F^0_{n,\omega} + \varepsilon\xi F^1_{n,\omega} + \varepsilon^2\xi^2 F^2_{n,\omega} + \cdots \tag{13b}$$

The governing differential equations of eigenvalue problems are obtained by setting q_n and $\gamma_n = 0$ in Eq. (8). The solution takes the form

$$\eta^0_n(\xi, \omega) = H^{(0)}_n(\xi, \omega) \cdot C_0 \tag{14a}$$

$$\eta^i_n(\xi, \omega) = H^{(i)}_n(\xi, \omega) \cdot C_0 \tag{14b}$$

where

$$H^{(0)}_n(\xi, \omega) = \exp\left(F^0_{n,c} + \omega^2 F^0_{n,\omega}\right)\xi$$

$$H^{(i)}_n(\xi, \omega) = H^{(0)}_n(\xi, \omega) \cdot \int_0^\xi \left[H^{(0)}_n(x, \omega)\right]^{-1} \times \sum_{j=1}^{i} x^j \left(F^j_{n,c} + \omega^2 F^j_{n,\omega}\right) \cdot H^{(i-j)}_n(x, \omega)\, \mathrm{d}x$$

and C_0 is a constant vector. The mth-order perturbation solution of $\eta_n(\xi, \omega)$ is written as

$$\eta_n(\xi, \omega) = \sum_{i=0}^{m} \varepsilon^i \eta^i_n(\xi, \omega) = \sum_{i=0}^{m} \varepsilon^i H^{(i)}_n(\xi, \omega) \cdot C_0 \tag{15}$$

Substituting Eq. (15) into Eq. (8b), we have following eigenvalue equations:

$$\det\left(M_n \cdot \sum_{i=0}^{m} \varepsilon^i H^{(i)}_n(-1, \omega_k) + N_n \cdot \sum_{i=0}^{m} \varepsilon^i H^{(i)}_n(1, \omega_k)\right) = 0 \qquad n = 1, 2, \ldots \tag{16a}$$

The corresponding eigenfunctions are given by

$$\eta_n(\xi, \omega_k) = \left(\sum_{i=0}^{m} \varepsilon^i H^{(i)}_n(\xi, \omega_k)\right) \cdot \psi_k, \qquad \xi \in (-1, 1) \tag{16b}$$

where the complex vector ψ_k is the nonzero solution of the following equation:

$$\left(M_n(s) \cdot \sum_{i=0}^{m} \varepsilon^i H^{(i)}_n(-1, \omega_k) + N_n(s) \cdot \sum_{i=0}^{m} \varepsilon^i H^{(i)}_n(1, \omega_k)\right) \cdot \psi_k = \mathbf{0} \tag{16c}$$

The eigenfunction vector $\eta_n(\xi, \omega_k)$ in Eq. (16b) simultaneously presents the modal distributions of the shell displacements and internal forces.

Combined Conical Shells

In engineering practice, more complicated cases often arise where a combined shell structure has several segments, as shown in Fig. 1. The formulations given in the preceding two sections cannot be used directly. Some examples are 1) the combined shell where each segment has a different conical angle; 2) the combined shell where each segment has a different thickness; and 3) there are other supports besides at the ends of the shell. Moreover, when the conical angle α is too large or the generatrix length L is too long so that ε is not a small enough parameter, one needs to look for ways to raise the precision and efficiency of the perturbation solution. One way is to cut the shell into several segments so that, within each subshell, ε is small enough.

In this section, combined conical shells are discussed. First, a combined conical shell is divided into conical shell elements. In each element, it is supposed that the conical angle α and shell thickness h are constants, there are no supports except at the ends of the element, and ε is a small enough perturbation parameter, although it may be different from element to element.

Within each subshell element, the perturbation method used in the last two sections is valid. Taking the mth-order perturbation solution as the asymptotic solution of the problem, Eq. (9b) is rewritten to

$$\eta_n(\xi, s) = H^m_n(\xi, s) \cdot \gamma_n(s) + f^m_n \tag{17}$$

where $H^m_n(\xi, s)$ and f^m_n are given by the recurrence formula

$$H^m_n(\xi, s) = \sum_{i=0}^{n} \varepsilon^i H_{ni}(\xi, s), \qquad f^m_n(\xi, s) = \sum_{i=0}^{n} \varepsilon^i f_{ni}(\xi, s)$$

$$H_{n,0}(\xi, s) = H_n(\xi, s), \qquad f_{n,0}(\xi, s) = \int_{-1}^{1} G_n(\xi, x, s) \cdot q_n(x)\, \mathrm{d}x$$

$$H_{n,i}(\xi, s) = \int_{-1}^{1} G_n(\xi, x, s) \cdot \sum_{j=0}^{i-1} x^{i-j} F^{i-j}_n \cdot H_{n,j}(x, s)\, \mathrm{d}x$$

$$f_{n,i}(\xi, s) = \int_{-1}^{1} G_n(\xi, x, s) \cdot \sum_{j=0}^{i-1} x^{i-j} F^{i-j}_n \cdot f_{n,j}(x, s)\, \mathrm{d}x$$

Taking the nodal displacement vector of the subshell as its boundary condition vector γ_n

$$\gamma^k_n = \begin{Bmatrix} \gamma^k_{1,n} \\ \gamma^k_{2,n} \end{Bmatrix} \tag{18}$$

where superscript k denotes the kth subshell element, $\gamma^k_{1,n}$ and $\gamma^k_{2,n}$ are the nodal displacement vectors of the left and right ends of the shell, respectively,

$$\gamma^k_{i,n} = \left\{\gamma^k_{1,1,n} \quad \gamma^k_{2,1,n} \quad \gamma^k_{3,1,n} \quad \gamma^k_{1,2,n} \quad \gamma^k_{2,2,n} \quad \gamma^k_{3,2,n}\right\}^T \Big|_{\xi=\xi_i} \qquad (i = 1, 2) \tag{19a}$$

$$\gamma^T_{i,j,n} = \left\{\tilde{u}^i_{j,n} \quad \frac{\mathrm{d}\tilde{u}^i_{j,n}}{\mathrm{d}\xi} \quad \cdots \quad \frac{\mathrm{d}^{[n_i/2]-1}\tilde{u}^i_{j,n}}{\mathrm{d}\xi^{[n_i/2]-1}}\right\}_{\xi=\xi_p} \qquad (i = 1, 2, 3,\ j = 1, 2,\ p = 1, 2) \tag{19b}$$

in which $\xi_1 = -1$ and $\xi_2 = 1$ are the ends of the shell and $[x]$ is the integer part of x.

The transform between global and local nodal displacement vectors is written as

$$\gamma^k_{n,1} = T_{k,L} \cdot \delta_{n,k} \tag{20a}$$

$$\gamma^k_{n,2} = T_{k,R} \cdot \delta_{n,k+1} \tag{20b}$$

where $T_{k,L}$ and $T_{k,R}$ are the transform matrices of the left and right ends and $\delta_{n,k}$ and $\delta_{n,k+1}$ the global displacement vectors of the left and right ends of the kth subshell.

The stress-displacement relation in the local coordinate is written as

$$\sigma^k_n = S^k \cdot \eta_n(\xi, s) \tag{21}$$

where S^k is the stiffness matrix of the kth subshell.

Substitute Eq. (17) into Eq. (21):

$$\sigma^k_n = k^k_L(\xi, s) \cdot T_{k,L} \cdot \delta_{n,k} + k^k_R(\xi, s) \cdot T_{k,R} \cdot \delta_{n,k+1} + q^k_n(\xi, s) \tag{22}$$

in which

$$[k_L^k \quad k_R^k] = S^k \cdot H_{n,k}^m(\xi, s)$$

$$q_n^k(\xi, s) = S^k \cdot f_n^m(\xi, s)$$

The force equilibrium at node k requires

$$T_{k,L}^T \cdot \sigma^k(-1, s) - T_{k-1,L}^T \cdot \sigma^{k-1}(1, s) = Q^k \qquad (k = 2, 3, \ldots, NE) \quad (23)$$

where Q^k is the concentrated load vector at node k and NE is the number of subshells.

Inserting Eq. (22) into Eq. (23) gives

$$-T_{k-1,R}^T \cdot k_L^{k-1}(1, s) \cdot T_{k-1,L} \cdot \delta_{n,k-1} + \left(T_{k,L}^T \cdot k_L^k(-1, s) \cdot T_{k,L} - T_{k-1,R}^T \cdot k_R^{k-1}(1, s) \cdot T_{k-1,R}\right) \cdot \delta_{n,k} + T_{k,L}^T \cdot k_R^k(-1, s) \times T_{k,R} \cdot \delta_{n,k+1} = Q^k - T_{k,L}^T \cdot q^k(-1, s) + T_{k-1,R}^T \cdot q^{k-1}(1, s) \qquad (k = 2, 3, \ldots, NE) \quad (24a)$$

At the left and right ends of the shell, the nodal force equilibrium equations are

$$T_{1,L}^T \cdot k_L^1(-1, s) \cdot T_{1,L} \cdot \delta_{n,1} + T_{1,L}^T \cdot k_R^1(-1, s) \cdot T_{1,R} \cdot \delta_{n,2} = Q^1 - T_{1,L}^T \cdot q^1(-1, s) \quad (24b)$$

$$T_{NE,R}^T \cdot k_L^{NE}(1, s) \cdot T_{NE,L} \cdot \delta_{n,NE} + T_{NE,R}^T \cdot k_R^{NE}(1, s) \times T_{NE,R} \cdot \delta_{n,NE+1} = -Q^{NE+1} - T_{NE,R}^T \cdot q^{NE}(1, s) \quad (24c)$$

The nodal force equilibrium equations (24) are assembled in the standard way to form the global equilibrium equation

$$K \cdot \delta_n = Q \quad (25)$$

where K is the global stiffness matrix and δ_n and Q are global nodal displacement and load vectors, respectively.

After introducing boundary conditions, the linear algebraic equation (25) can be solved easily. Using Eq. (17), the displacements and stresses at any position of the shell can be found.

For eigenvalue problems of the combined conical shell $Q = 0$, the eigenvalues are determined from $\det K = 0$.

Application to Love–Timoshenko Shell Equations

From the preceding derivation, it is easy to see that the method can be used for analysis of conical shells regardless of what kinds of linear shell motion equations are adopted. As an application of the developed theory, the static response and free vibration of conical shells using Love–Timoshenko shell equations are studied. Without loss of generality, the shell is supposed to be homogeneous, isotropic, and elastic. The strain-displacement relation is[13]

$$\varepsilon_x^0 = \frac{\partial u}{\partial x} \quad (26a)$$

$$\varepsilon_\theta^0 = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\sin\alpha}{r}u + \frac{\cos\alpha}{r}w \quad (26b)$$

$$\varepsilon_{x\theta}^0 = \frac{\partial v}{\partial x} - \frac{\sin\alpha}{r}v + \frac{1}{r}\frac{\partial u}{\partial \theta} \quad (26c)$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \quad (26d)$$

$$\kappa_\theta = \frac{\cos\alpha}{r}\frac{\partial v}{\partial \theta} - \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{\sin\alpha}{r}\frac{\partial w}{\partial x} \quad (26e)$$

$$\kappa_{x\theta} = -\frac{2\sin\alpha\cos\alpha}{r^2}v + \frac{\cos\alpha}{r}\frac{\partial v}{\partial x} + \frac{2\sin\alpha}{r^2}\frac{\partial w}{\partial \theta} - \frac{2}{r}\frac{\partial^2 w}{\partial x\,\partial \theta} \quad (26f)$$

and the stress-strain relation takes the form[13]

$$N_x = K(\varepsilon_x + \mu\varepsilon_\theta) \quad (27a)$$

$$N_\theta = K(\varepsilon_\theta + \mu\varepsilon_x) \quad (27b)$$

$$N_{x\theta} = N_{\theta x} = \frac{K(1-\mu)}{2}\varepsilon_{x\theta} \quad (27c)$$

$$M_x = D(\kappa_x + \mu\kappa_\theta) \quad (27d)$$

$$M_\theta = D(\kappa_\theta + \mu\kappa_x) \quad (27e)$$

$$M_{x\theta} = M_{\theta x} = \frac{D(1-\mu)}{2}\kappa_{x\theta} \quad (27f)$$

where μ is Poisson's ratio and K and D are tension and bending stiffness

$$K = Eh/(1-\mu^2), \qquad D = Eh^3/(1-\mu^2) \quad (28)$$

where E is Young's modulus and h is the thickness of the shell.

The equations of motion are[13]

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{x\theta}}{\partial \theta} + \frac{\sin\alpha}{r}(N_x - N_\theta) + q_x = \rho h\frac{\partial^2 u}{\partial t^2} \quad (29a)$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{2\sin\alpha}{r}N_{\theta x} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{\cos\alpha}{r}Q_\theta + q_\theta = \rho h\frac{\partial^2 v}{\partial t^2} \quad (29b)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\sin\alpha}{r}Q_x + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} - \frac{\cos\alpha}{r}N_\theta + q_z = \rho h\frac{\partial^2 w}{\partial t^2} \quad (29c)$$

in which Q_x and Q_θ are shear forces defined as

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\sin\alpha}{r}M_x + \frac{1}{r}\frac{\partial M_{\theta x}}{\partial \theta} - \frac{\sin\alpha}{r}M_\theta$$

$$Q_\theta = \frac{\partial M_{x\theta}}{\partial x} + \frac{2\sin\alpha}{r}M_{\theta x} + \frac{1}{r}\frac{\partial M_\theta}{\partial \theta}$$

The four boundary conditions are

$$u(x_i, \theta, t) = \bar{u}_i(\theta, t) \quad \text{or} \quad N_x(x_i, \theta, t) = \bar{N}_{x,i}(\theta, t) \quad (30a)$$

$$v(x_i, \theta, t) = \bar{v}_i(\theta, t) \quad \text{or} \quad T_{x\theta}(x_i, \theta, t) = \bar{T}_{x\theta,i}(\theta, t) = \bar{N}_{x\theta,i}(\theta, t) + (\cos\alpha/r)\bar{M}_{x\theta,i}(\theta, t) \quad (30b)$$

$$w(x_i, \theta, t) = \bar{w}_i(\theta, t) \quad \text{or} \quad V_x(x_i, \theta, t) = \bar{V}_{x,i}(\theta, t) = \bar{Q}_{x,i}(\theta, t) + \frac{1}{r}\frac{\partial \bar{M}_{x\theta,i}(\theta, t)}{\partial \theta} \quad (30c)$$

$$\left.\frac{\partial w(x, \theta, t)}{\partial x}\right|_{x=x_i} = \bar{\psi}_i(\theta, t) \quad \text{or} \quad M_x(x_i, \theta, t) = \bar{M}_{x,i}(\theta, t) \quad (30d)$$

where x_i $(i = 1, 2)$ are the x coordinates of the left and right ends of the shell and $\bar{u}_i$, $\bar{v}_i$, $\bar{w}_i$, $\bar{\psi}_i$, $\bar{N}_{x,i}$, $\bar{N}_{x\theta,i}$, $\bar{Q}_{x,i}$, $\bar{M}_{x,i}$, and $\bar{M}_{x\theta,i}$ $(i = 1, 2)$ are prescribed functions.

To make statements simpler and without loss of generality, only the symmetric parts in the Fourier expansion are considered in the following. Hence,

$$u(x, \theta, t) = \sum_n u_n(x, t)\cos n\theta \quad (31a)$$

$$v(x, \theta, t) = \sum_n v_n(x, t)\sin n\theta \quad (31b)$$

$$w(x, \theta, t) = \sum_n w_n(x, t)\cos n\theta \quad (31c)$$

$$q_x(x, \theta, t) = \sum_n q_{x,n}(x, t)\cos n\theta \quad (31d)$$

$$q_\theta(x, \theta, t) = \sum_n q_{\theta,n}(x, t)\sin n\theta \quad (31e)$$

$$q_z(x,\theta,t)=\sum_n q_{z,n}(x,t)\cos n\theta \tag{31f}$$

$$\bar{u}_i(\theta,t)=\sum_n \bar{u}_{i,n}(t)\cos n\theta \tag{31g}$$

$$\bar{v}_i(\theta,t)=\sum_n \bar{v}_{i,n}(t)\sin n\theta \tag{31h}$$

$$\bar{w}_i(\theta,t)=\sum_n \bar{w}_{i,n}(t)\cos n\theta \tag{31i}$$

$$\bar{N}_{x,i}(\theta,t)=\sum_n \bar{N}_{x,i,n}(t)\cos n\theta \tag{31j}$$

$$\bar{N}_{x\theta,i}(\theta,t)=\sum_n \bar{N}_{x\theta,i,n}(t)\sin n\theta \tag{31k}$$

$$\bar{M}_{x\theta,i}(\theta,t)=\sum_n \bar{M}_{x\theta,i,n}(t)\sin n\theta \tag{31l}$$

$$\bar{Q}_{x,i}(\theta,t)=\sum_n \bar{Q}_{x,i,n}(t)\cos n\theta \tag{31m}$$

$$\bar{M}_{x,i}(\theta,t)=\sum_n \bar{M}_{x,i,n}(t)\cos n\theta \tag{31n}$$

Substituting Eqs. (31) into Eqs. (29) and (30), decoupled differential equations of equilibrium and boundary conditions are derived. For the sake of simplicity, the subscript n is omitted in the following. The state-space form of the differential equations of motion are

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}\xi}=\boldsymbol{F}(\varepsilon\xi,s)\cdot\boldsymbol{\eta}+\boldsymbol{q}(\xi,s) \tag{32}$$

in which the state-space vector $\boldsymbol{\eta}$ and the nonzero elements of $\boldsymbol{F}$ and $\boldsymbol{q}$ are

$$\boldsymbol{\eta}(x,s)=\{u \quad u' \quad v \quad v' \quad w \quad w' \quad w'' \quad w'''\}^T$$

$$F_{12}=1,\qquad F_{21}=a_2(1+\varepsilon\xi)^{-2}+a_3$$

$$F_{22}=a_1(1+\varepsilon\xi)^{-1},\qquad F_{23}=a_4(1+\varepsilon\xi)^{-2}$$

$$F_{24}=a_5(1+\varepsilon\xi)^{-1},\qquad F_{25}=a_6(1+\varepsilon\xi)^{-2}$$

$$F_{26}=a_7(1+\varepsilon\xi)^{-1},\qquad F_{34}=1$$

$$g_1(\varepsilon\xi)=\left(1+2b_3\varepsilon\xi+b_3\varepsilon^2\xi^2\right)^{-1},\qquad F_{41}=b_1g_1(\varepsilon\xi)$$

$$F_{42}=b_2(1+\varepsilon\xi)^{-1}g_1(\varepsilon\xi)$$

$$F_{43}=\left[b_6(1+\varepsilon\xi)^{-2}+b_7+b_8(1+\varepsilon\xi)^2\right]g_1(\varepsilon\xi)$$

$$F_{44}=\left[b_4(1+\varepsilon\xi)+b_5(1+\varepsilon\xi)^{-1}\right]g_1(\varepsilon\xi)$$

$$F_{45}=\left[b_{11}+b_{12}(1+\varepsilon\xi)^{-2}\right]g_1(\varepsilon\xi)$$

$$F_{46}=b_{10}(1+\varepsilon\xi)^{-1}g_1(\varepsilon\xi),\qquad F_{47}=b_9g_1(\varepsilon\xi)$$

$$F_{56}=F_{67}=F_{78}=1,\qquad g_2(\varepsilon\xi)=c_3g_1(\varepsilon\xi)(1+\varepsilon\xi)^{-1}$$

$$F_{81}=c_2(1+\varepsilon\xi)^{-2}+b_1g_2(\varepsilon\xi)$$

$$F_{82}=c_1(1+\varepsilon\xi)^{-1}+b_2(1+\varepsilon\xi)g_2(\varepsilon\xi)$$

$$F_{83}=c_5(1+\varepsilon\xi)^{-4}+\left[b_6(1+\varepsilon\xi)^{-2}+b_7+b_8(1+\varepsilon\xi)^2\right]$$
$$\times g_2(\varepsilon\xi)+c_5'(1+\varepsilon\xi)^{-2}$$

$$F_{84}=c_4(1+\varepsilon\xi)^{-3}+\left[b_4(1+\varepsilon\xi)+b_5(1+\varepsilon\xi)^{-1}\right]g_2(\varepsilon\xi)$$

$$F_{85}=c_9(1+\varepsilon\xi)^{-4}+\left[b_{11}+b_{12}(1+\varepsilon\xi)^{-2}\right]g_2(\varepsilon\xi)$$
$$+c_{10}(1+\varepsilon\xi)^{-2}+c_{11}$$

$$F_{86}=c_8(1+\varepsilon\xi)^{-3}+b_{10}(1+\varepsilon\xi)^{-1}g_2(\varepsilon\xi)$$

$$F_{87}=c_7(1+\varepsilon\xi)^{-2}+b_9g_2(\varepsilon\xi)$$

$$F_{88}=c_6(1+\varepsilon\xi)^{-1},\qquad q_2=-\left(q_xL^2/K\right)$$

$$q_4=-\left(q_\theta L^2/b_0K\right)(1+\varepsilon\xi)^2g_1(\varepsilon\xi)$$

$$q_8=\left(q_zL^4/D\right)-\left(q_\theta L^2/b_0K\right)c_3g_2(\varepsilon\xi)$$

and the coefficients a_i, b_j, and c_k are defined as

$$a_1=-\lambda\sin\alpha,\qquad a_2=\lambda^2\left(\frac{1-\mu}{2}n^2+\sin^2\alpha\right)$$

$$a_3=\lambda^2\frac{\rho hs^2r_0^2}{K},\qquad a_4=\frac{n}{2}(3-\mu)\lambda^2\sin\alpha$$

$$a_5=-\frac{n\lambda}{2}(1+\mu),\qquad a_6=\lambda^2\sin\alpha\cos\alpha,\qquad a_7=-\lambda\mu\cos\alpha$$

$$b_0=\frac{1-\mu}{2r_0^2}\left(r_0^2+\beta\cos^2\alpha\right),\qquad b_1=\frac{\lambda^2n(3-\mu)\sin\alpha}{2b_0}$$

$$b_2=\frac{(1+\mu)n\lambda}{2b_0},\qquad b_3=\frac{(1-\mu)}{2b_0}$$

$$b_4=-\frac{\lambda(1-\mu)\sin\alpha}{2b_0},\qquad b_5=\frac{\lambda\beta(1-\mu)\sin\alpha\cos^2\alpha}{2b_0r_0^2}$$

$$b_6=\frac{\lambda^2n^2\beta\cos^2\alpha}{b_0r_0^2},\qquad b_7=\frac{\lambda^2}{b_0}\left(n^2+\frac{1-\mu}{2}\sin^2\alpha\right)$$

$$b_8=\frac{\lambda^2r_0^2\rho hs^2}{Kb_0},\qquad b_9=-\frac{n\beta\cos\alpha}{b_0r_0^2}$$

$$b_{10}=-\frac{n\beta\lambda\sin 2\alpha}{2b_0r_0^2},\qquad b_{11}=\frac{n\lambda^2\cos\alpha}{b_0}$$

$$b_{12}=\frac{\beta\lambda^2n^3\cos\alpha}{b_0r_0^2},\qquad c_1=-\beta^{-1}\lambda\mu L^2\cos\alpha$$

$$c_2=-\beta^{-1}\lambda^2L^2\sin\alpha\cos\alpha,\qquad c_3=n\lambda^2\cos\alpha$$

$$c_4=-3n\lambda^3\sin\alpha\cos\alpha,\qquad c_5=n\lambda^4(4\sin^2\alpha-n^2)\cos\alpha$$

$$c_6=-2\lambda\sin\alpha,\qquad c_7=\lambda^2(2n^2+\sin^2\alpha)$$

$$c_8=-\lambda^3(2n^2+\sin^2\alpha)\sin\alpha,\qquad c_9=-\lambda^4(n^4-4n^2\sin^2\alpha)$$

$$c_{10}=-\beta^{-1}\lambda^4r_0^2\cos^2\alpha,\qquad c_{11}=-\frac{\lambda^4r_0^4\rho hs^2}{D}$$

$$c_5'=-\beta^{-1}n\lambda^4r_0^2\cos\alpha,\qquad \beta=\frac{D}{K}$$

The boundary condition matrices are

$$\boldsymbol{M}=\begin{bmatrix}\boldsymbol{B}^L\\ 0_{4\times8}\end{bmatrix}\cdot\boldsymbol{B}\bigg|_{x=x_1},\qquad \boldsymbol{N}=\begin{bmatrix}0_{4\times8}\\ \boldsymbol{B}^R\end{bmatrix}\cdot\boldsymbol{B}\bigg|_{x=x_2}$$

$$\gamma=\begin{bmatrix}\boldsymbol{B}^L\cdot\bar{\gamma}_1\\ \boldsymbol{B}^R\cdot\bar{\gamma}_2\end{bmatrix}$$

where the nonzero elements of $\boldsymbol{B}$ are

$$B_{11}=B_{33}=B_{55}=1,\qquad B_{21}=\frac{\mu}{r}\sin\alpha$$

$$B_{22}=\frac{1}{L},\qquad B_{23}=\frac{\mu n}{r},\qquad B_{25}=\frac{\mu}{r}\cos\alpha$$

$$B_{41}=-\frac{n}{r},\qquad B_{43}=-\frac{\sin\alpha}{r}\left(1+\frac{2\beta\cos^2\alpha}{r^2}\right)$$

$$B_{44} = \frac{1}{L}\left(1 + \frac{\beta\cos^2\alpha}{r^2}\right), \qquad B_{45} = -\frac{n\beta}{r^3}\sin 2\alpha$$

$$B_{46} = \frac{2n\beta}{r^2 L}\cos\alpha, \qquad B_{63} = \frac{(-3+\mu)n}{2r^3}\sin 2\alpha$$

$$B_{64} = \frac{n}{r^2 L}\cos\alpha, \qquad B_{65} = \frac{(-3+\mu)n^2}{r^3}\sin\alpha$$

$$B_{66} = \frac{1}{r^2 L}(\sin^2\alpha + 2n^2 - \mu n^2), \qquad B_{67} = -\frac{1}{L^2 r}\sin\alpha$$

$$B_{68} = -\frac{1}{L^2}, \qquad B_{76} = \frac{1}{L}, \qquad B_{83} = \frac{n\mu}{r^2}\cos\alpha$$

$$B_{86} = -\frac{\mu\sin\alpha}{rL^2}, \qquad B_{87} = -\frac{1}{L^3}$$

and $\bar{\gamma}_i$ $(i = 1, 2)$ are the boundary condition vectors including the displacements and forces defined on the boundary:

$$\bar{\gamma}_i = \left\{ \bar{u}_i \quad \frac{\bar{N}_{x,i}}{K} \quad \bar{v}_i \quad \frac{2\bar{T}_{x\theta,i}}{k(1-\mu)} \quad \bar{w}_i \quad \frac{\bar{V}_{x,i}}{D} \quad \bar{\psi}_i \quad \frac{\bar{M}_{x,i}}{D} \right\}^T$$

In $\bar{\gamma}_i$, only half of its elements are prescribed. Those unknowns are simply taken to be zero because the corresponding columns in $\boldsymbol{B}^L$ or $\boldsymbol{B}^R$ are zero. $\boldsymbol{B}^L$ is the selective matrix of boundary conditions at the left end of the shell. Its ith row is corresponding to the ith boundary condition defined by Eq. (30). If it is a displacement condition, then

$$B^L_{i,2i-1} = 1, \qquad B^L_{ij} = 0 \qquad (j \neq 2i-1)$$

Otherwise

$$B^L_{i,2i} = 1, \qquad B^L_{ij} = 0 \qquad (j \neq 2i)$$

$\boldsymbol{B}^R$ has the same definition as $\boldsymbol{B}^L$ except that it is related to the right end of the subshell.

In the analysis of combined conical shells, the global nodal displacement vector is simplified to

$$\delta_i = \left\{ u_i \quad v_i \quad w_i \quad \frac{\mathrm{d}w_i}{\mathrm{d}x} \right\}^T \qquad (i = 1, 2, \ldots, NE+1) \tag{33}$$

$\boldsymbol{M}$ and $\boldsymbol{N}$ have the form

$$\boldsymbol{M} = \begin{bmatrix} \bar{\boldsymbol{B}} \\ 0_{4\times 8} \end{bmatrix}, \qquad \boldsymbol{N} = \begin{bmatrix} 0_{4\times 8} \\ \bar{\boldsymbol{B}} \end{bmatrix}$$

where the nonzero elements of $\bar{\boldsymbol{B}}$ are $\bar{B}_{11} = \bar{B}_{23} = \bar{B}_{35} = \bar{B}_{46} = 1$.

Numerical Results

Static Response of Conical Shell

The first example (Fig. 3) is the static response of a conical shell subjected to inner pressure. The material and geometric parameters of the shell are chosen to be $E = 10^5$, $\mu = 0.3$, $\alpha = 10$ deg, $r_2 = 100$, $L = 50$, $h = 1$, $\rho = 1$, and $q = 100$.

A second-order perturbation solution is adopted in numerical simulations. The shell is not divided into subshells. As a comparison, a finite element method (FEM) solution using conical shell elements is obtained. Numerical simulation results are shown in Fig. 4.

Numerical results show that, to obtain accurate enough results, FEM has to use a lot of elements, especially when stress concentration exists, such as the clamped end. In this example, to satisfy precision requirement of displacements, 50 elements are needed. The same precision requirement for stress costs 250 elements. To show the convergence tendency, a comparison of our transfer function method (TFM) and FEM using different element numbers is shown in Fig. 5.

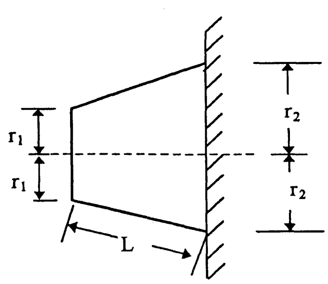

Fig. 3 Free-clamped conical shell.

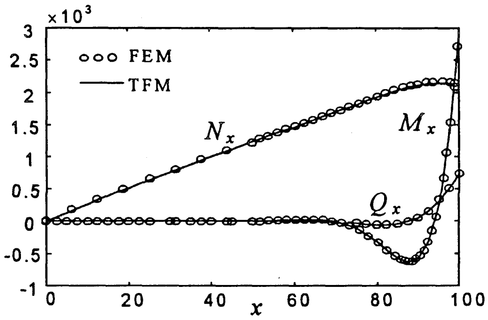

Fig. 4 Stress of free-clamped conical shell subjected to inner pressure.

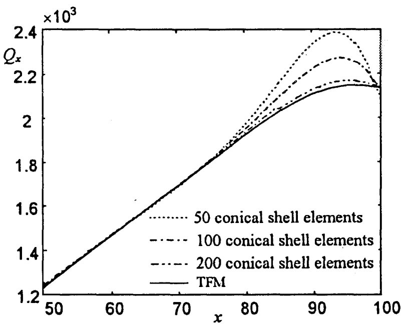

Fig. 5a Shear stresses of free-clamped conical shell subjected to inner pressure.

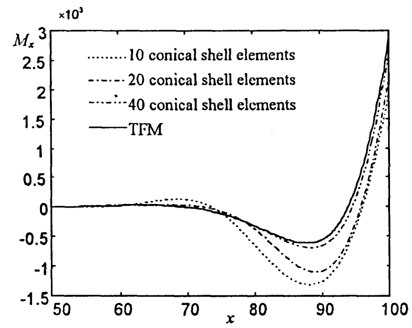

Fig. 5b Bending moments of free-clamped conical shell subjected to inner pressure.

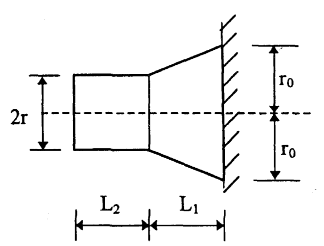

Fig. 6 Combined conical shell.

Static Response of Stepped Conical Shell

A combined conical shell consists of a cylindrical shell segment and a conical shell segment as shown in Fig. 6 is analyzed. The material parameters of the shell are chosen as $E = 10^5$, $\mu = 0.3$, and $\rho = 1$; for the conical segment $\alpha = 10$ deg, $r_0 = 100$, $L_1 = 50\cos\alpha$, and $h = 1$; for the cylindrical segment $L_2 = 50$ and $h = 1$.

The shell is subjected to inner pressure $q = 100$. A third-order perturbation solution is obtained. Numerical results are shown in Fig. 7, where FEM uses 240 conical shell elements.

Table 1 Natural frequencies ω_{mn} of the free-clamped conical shell

	ω_{m3}		ω_{m4}		ω_{m5}		ω_{m6}		ω_{m7}	
m	TFM	FEM	TFM	FEM	TFM	FEM	TFM	FEM	TFM	FEM
1	0.7974	0.8023	0.5707	0.5733	0.4994	0.4995	0.5465	0.5465	0.6705	0.6706
2	2.1502	2.1396	1.6939	1.6888	1.3880	1.3906	1.2193	1.2210	1.1630	1.1637
3	2.9627	2.9617	2.6617	2.6689	2.3944	2.3971	2.1773	2.1799	2.0346	2.0361
4	3.3697	3.4089	3.2148	3.2500	3.0902	3.0920	2.9523	2.9544	2.8522	2.8535
5	3.8637	3.8485	3.8158	3.7834	3.7174	3.7189	3.6642	3.6656	3.6333	3.6342

Table 2 Natural frequencies ω_{mn} of the free-clamped combined conical shell

	ω_{m4}		ω_{m5}		ω_{m6}		ω_{m7}		ω_{m8}	
m	TFM	FEM	TFM	FEM	TFM	FEM	TFM	FEM	TFM	FEM
1	0.8117	0.8119	0.7422	0.7423	0.6960	0.6962	0.7405	0.7407	0.8616	0.8617
2	1.1698	1.1718	0.9080	0.9090	0.9166	0.9168	1.0543	1.0545	1.2420	1.2422
3	2.5773	2.5811	2.3099	2.3139	2.1005	2.1041	1.9579	1.9619	1.9027	1.9050
4	3.2261	3.2281	3.0538	3.0560	2.8966	2.8988	2.7793	2.7815	2.7190	2.7211
5	3.7308	3.7335	3.6750	3.6764	3.6263	3.6276	3.6263	3.5964	3.5936	3.5950

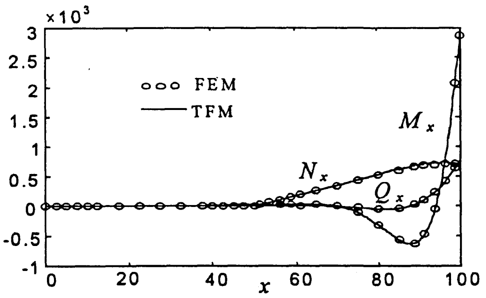

Fig. 7 Stress of free-clamped stepped conical shell subjected to inner pressure.

Free Vibration Frequency of Conical Shell

In this example, the material and geometric parameters and the boundary conditions of the shell are taken to be the same as that of the first and second examples. The first five free vibration frequencies are given in Tables 1 and 2, where m is the longitudinal wave number and n is the circumferential wave number.

Conclusions

A new asymptotic TFM is presented for the analysis of conical shells. Using a Fourier expansion, perturbation technique, and Laplace transform, the original partial differential equations are reduced to a series of decoupled ordinary differential equations with constant coefficients, which are solved by a state-space technique. By expressing the asymptotic solution of a shell segment in its nodal displacements and imposing force balance and displacement compatibility at each node, complex combined conical shells composed of different conical shell segments are analyzed.

The proposed asymptotic TFM provides a systematic way to study conical shells; the solution procedure is the same for different shell models, initial and boundary conditions, and external excitations. Therefore, the method is very convenient in computer coding and numerical simulation.

The numerical simulation on the static response and free vibration of the Love–Timoshenko shell shows that the method has very high precision. The efficiency of the method is much higher than that of the FEM, especially in the calculation of stress and strain. The reasons are twofold. First, the method gives an asymptotic solution of high precision within each shell segment. Second, the method obtains not only the displacements but also their derivatives, i.e., strains and stresses are obtained simultaneously from the solution of governing differential equations.

Note that the method presented can be incorporated into the FEM to solve more complicated initial boundary value problems. Both methods can use the same nodal displacement definition, and the solution procedures for a combined shell are the same except for the determination of shape functions. Because a very good asymptotic solution to the state-space vector is obtained in a closed form, it can be expected that, for a conical shell element, our method is more efficient than the finite element method.

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